A BLASCHKE-TYPE CONDITION AND ITS APPLICATION TO COMPLEX JACOBI MATRICES

A. BORICHEV, L. GOLINSKII, S. KUPIN

Abstract. We obtain a Blaschke-type necessary condition on zeros of analytic functions on the unit disk with different types of exponential growth at the boundary. These conditions are used to prove Lieb-Thirring-type inequalities for the eigenvalues of complex Jacobi matrices.

Introduction

In the first part of the paper, we obtain some information on the distribution of the zeros of analytic functions from special growth classes. We use this information to get interesting counterparts of famous Lieb-Thirring inequalities [8, 9] for complex Jacobi matrices.

The traditional approach to Lieb-Thirring bounds for complex Jacobi matrices consists in deducing them from the bounds for corresponding self-adjoint objects, see Frank-Laptev-Lieb-Seiringer [3] and Golinskii-Kupin [6]. However, this method is quite limited. At best, it allows us to get an information on a part of the point spectrum $\sigma_p(J)$ of a Jacobi matrix J, situated in a very special diamond-shaped region, see [6, Theorem 1.5]. The information on the whole $\sigma_p(J)$ is missing.

The main idea of our paper is to use functional-theoretic tools in the problem described above. Let $J = J(\{a_k\}, \{b_k\}, \{c_k\})$ be a complex Jacobi matrix (2.1) such that $J - J_0$ lies in the Schatten-von Neumann class $\mathcal{S}_p$, $p \geq 1$, and $J_0 = J(\{1\}, \{0\}, \{1\})$ (see Section 2 for terminology). For an integer p, the regularized perturbation determinant $u_p(\lambda) = \det_p(J - \lambda)(J_0 - \lambda)^{-1}$ is well-defined, and is an analytic function on $\hat{\mathbb{C}}\backslash\sigma(J_0) = \hat{\mathbb{C}}\backslash[-2, 2]$. Its zero set coincides with $\sigma_p(J)$ up to multiplicities. We can obtain some information on the distribution of the zeros of the function u_p from its growth estimates in a neighborhood of the boundary of $\hat{\mathbb{C}}\backslash[-2, 2]$. As usual, the domain $\hat{\mathbb{C}}\backslash[-2, 2]$ is mapped conformally to the unit disk $\mathbb{D}$, so we are mainly interested in properties of corresponding analytic functions on $\mathbb{D}$.

Let $\mathcal{A}(\mathbb{D})$ be the set of analytic functions on the unit disk $\mathbb{D}$, $f \in \mathcal{A}(\mathbb{D})$, $f \neq 0$, and $Z_f = \{z_j\}$ denote the set of zeros of f.

Date: December 1, 2007.
1991 *Mathematics Subject Classification.* Primary: 30C15; Secondary: 47B36.
Key words and phrases. Blaschke-type estimates, Lieb-Thirring inequalities for Jacobi matrices.

Theorem 0.1. *Given a finite set* $E = \{\zeta_j\}_{j=1,\dots,N}$*,* $E \subset \mathbb{T}$*, let* $f \in \mathcal{A}(\mathbb{D})$*,* $|f(0)| = 1$*, and*

$$|f(z)| \le \exp\Bigl(\frac{D}{\operatorname{dist}(z,E)^q}\Bigr), \tag{0.1}$$

with $q \ge 0$*. Then for any* $\varepsilon > 0$*,*

$$\sum_{z\in Z_f} (1-|z|)\operatorname{dist}(z,E)^{(q-1+\varepsilon)_+} \le C(\varepsilon,q,E)\,D. \tag{0.2}$$

Here, $x_+ = \max\{x,0\}$.

Clearly, (0.2) is a Blaschke-type condition. The classical Blascke condition is valid for functions from Hardy spaces $H^p(\mathbb{D}), 0 < p \le \infty$, or, more generally, from the Nevanlinna class $\mathcal{N}$, see Garnett [4, Ch. 2], and follows from the Poisson–Jensen formula:

$$\sum_{z\in Z_f} (1-|z|) \le \sup_r \int_{\mathbb{T}} \log|f(r\zeta)|\,dm - \log|f(0)|, \tag{0.3}$$

where dm is the normalized Lebesgue measure on $\mathbb{T}$. With small modifications one can write its analogs for the Bergman and the Korenblum spaces, see Hedenmalm-Korenblum-Zhu [7]. The specifics of our particular problem lead us to consider weights with a finite number of "exponential singularities" at the boundary in addition to the radial growth of the weight.

Note that for $q < 1$ the function f (0.1) is in $\mathcal{N}$, and (0.2) follows directly from (0.3). So in the proof of (0.2) we can assume $q \ge 1$.

Theorem 0.2. *Let* $f \in \mathcal{A}(\mathbb{D})$*,* $|f(0)| = 1$*, and*

$$|f(z)| \le \exp\Bigl(\frac{D_1}{(1-|z|)^p \operatorname{dist}(z,E)^q}\Bigr), \tag{0.4}$$

where $p, q \ge 0$*. Then for any* $\varepsilon > 0$*,*

$$\sum_{z\in Z_f} (1-|z|)^{p+1+\varepsilon}\operatorname{dist}(z,E)^{(q-1+\varepsilon)_+} \le C(\varepsilon,p,q,E)\,D_1. \tag{0.5}$$

One can vary the degrees of the singularities ζ_j in (0.1) and (0.4). More precisely,

Theorem 0.3. *Let*

$$h_1(z) = \left(\prod_{j=1}^N |z-\zeta_j|^{q_j}\right)^{-1}, \qquad h_2(z) = \left((1-|z|)^p \prod_{j=1}^N |z-\zeta_j|^{q_j}\right)^{-1},$$

with $\{q_j\}_{j=1,\dots,N}$*,* $q_j \ge 0$*. Furthermore, let* $f_j \in \mathcal{A}(\mathbb{D})$*,* $|f_j(0)| = 1$*,* $j = 1, 2$*, satisfy*

$$|f_j(z)| \le \exp(K_j h_j(z)).$$

Then

$$\sum_{z\in Z_{f_1}} (1-|z|) \prod_{j=1}^{N} |z-w_j|^{(q_j-1+\varepsilon)_+} \leq C(\varepsilon, \{q_j\}, E) K_1, \tag{0.6}$$

$$\sum_{z\in Z_{f_2}} (1-|z|)^{p+1+\varepsilon} \prod_{j=1}^{N} |z-w_j|^{(q_j-1+\varepsilon)_+} \leq C(\varepsilon, p, \{q_j\}, E) K_2. \tag{0.7}$$

For the sake of simplicity, we prove relations (0.2), (0.5). The proofs of (0.6), (0.7) are similar.

The starting idea of the work is close in spirit to an interesting paper by Demuth-Katriel [1]. To obtain counterparts of Lieb-Thirring bounds, the authors look at the difference of two semigroups generated by two continuous Schrödinger operators and they apply the classical Poisson-Jensen formula. They work with nuclear and Hilbert-Schmidt perturbations and the potential has to be from the Kato class. Our methods seem to be more straightforward. The computations are simpler and they are valid for $\mathcal{S}_p$-perturbations, $p \geq 1$. In particular, we do not require the *self-adjointness* of the perturbed operator.

As usual, $\mathbb{D}_r = \{z : |z| < r\}$, $\mathbb{D} = \mathbb{D}_1$, $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{T} = \mathbb{T}_1$, and $B(z_0, \delta) = \{z : |z - z_0| < \delta\}$. C is a constant changing from one relation to another one.

We proceed as follows. In Section 2 we prove Theorem 0.1 and derive Theorem 0.2 from it. In Section 3 we discuss applications to complex Jacobi matrices.

1. Proofs of Theorems 0.1 and 0.2

Given a circular arc $\Gamma = [e^{it_1}, e^{it_2}]$, denote by $\omega(z; t_1, t_2)$ its harmonic measure with respect to the unit disk $\mathbb{D}$. An explicit formula is available (see Garnett [4, Ch. 1, Exercise 3])

$$\omega(z; t_1, t_2) = \frac{1}{\pi}\left(\alpha(z) - \frac{t_1 - t_2}{2}\right),$$

where Γ is seen from a point z under the angle α. We use the notation ω_γ for the symmetric arc

$$\Gamma = \{\zeta \in \mathbb{T} : |1-\zeta| \leq \gamma\} = [e^{-it(\gamma)}, e^{it(\gamma)}], \quad \sin\frac{t(\gamma)}{2} = \frac{\gamma}{2}.$$

Here $\gamma = \gamma(E) < 1/500N$ is a small parameter which will depend on E. By the Mean Value Theorem

$$\frac{\gamma}{\pi} \leq \omega_\gamma(0) = \frac{t(\gamma)}{\pi} \leq \frac{\gamma}{2}. \tag{1.1}$$

Next,

$$\omega_\gamma(z) \geq \frac{1}{2} - \frac{t(\gamma)}{\pi} \geq \frac{1}{4} \tag{1.2}$$

for $|1-z|\le\gamma$. An outer function

$$g_\gamma(z)=e^{\omega_\gamma(z)+i\widetilde{\omega_\gamma(z)}},\quad |g_\gamma(\zeta)|=\begin{cases} e, & |1-\zeta|\le\gamma,\\ 1, & |1-\zeta|>\gamma,\end{cases} \tag{1.3}$$

where $\zeta\in\mathbb{T}$, will play a key role in what follows. Clearly, $\omega_\gamma=\log|g_\gamma|$, and

$$1\le|g_\gamma(z)|\le e, \tag{1.4}$$

for $z\in\mathbb{D}$.

We need a bound for the Blaschke product

$$b_\lambda(z):=\frac{z-\lambda}{1-\bar\lambda z}$$

in the case when the parameter λ and variable z are "well-separated".

Lemma 1.1. *Let $M=200N$. Then for $|1-z|=\gamma$, $|1-\lambda|\ge M\gamma$ one has*

$$\log\frac{1}{|b_\lambda(z)|}\le\frac{1}{4N\gamma}\,\log\frac{1}{|\lambda|}\,. \tag{1.5}$$

Proof. Obviously, we asume $\lambda\neq0$. Consider the domain

$$U:=\mathbb{D}\backslash B(\lambda,r),\quad r=\frac{1-|\lambda|}{4}\,,$$

and two harmonic functions on U

$$V_\lambda(z):=\log\frac{1}{|b_\lambda(z)|},\quad W_\lambda(z):=\frac{(1+\varepsilon)^2-|z|^2}{|(1+\varepsilon)w-z|^2}\,,$$

where parameters $\varepsilon=\varepsilon(\lambda)>0$ and $w=w(\lambda)\in\mathbb{T}$ are chosen later on. Clearly, $V_\lambda(\zeta)=0<W_\lambda(\zeta)$ for $\zeta\in\mathbb{T}$, and we want to bound these functions on $\partial B(\lambda,r)$.

This is easy for V_λ:

$$\frac{1}{|b_\lambda(z)|}=\frac{4}{1-|\lambda|}\,|1-\bar\lambda z|,$$

and for $z=\lambda+re^{it}$ we have

$$1-\bar\lambda z=1-|\lambda|^2-\frac{1-|\lambda|}{4}\,\bar\lambda e^{it}$$

so $|1-\bar\lambda z|<3(1-|\lambda|)$ and

$$V_\lambda(z)<\log30<5,$$

where $z\in\partial B(\lambda,r)$.

The problem is more delicate for the lower bound on W_λ. Let $\lambda=|\lambda|e^{i\theta}$ and $w=e^{i\varphi}$. Then, for $z\in\partial B(\lambda,r)$

$$(1+\varepsilon)^2-|z|^2>1-|z|^2=1-|\lambda|^2-r^2-2r|\lambda|\cos(\theta-t)>\frac13(1-|\lambda|). \tag{1.6}$$

Next, we want to have a bound $O((1-|\lambda|)^2)$ for the denominator of W_λ

$$|(1+\varepsilon)w-z|^2 = |w-z|^2+\varepsilon^2+2\varepsilon\mathrm{Re}\,(1-\bar{w}z). \tag{1.7}$$

For the first term,

$$\begin{aligned}|w-z|^2 &= 1-2\mathrm{Re}\,\bar{w}z+|\lambda|^2+r^2+2r|\lambda|\cos(\theta-t) = S_1+S_2,\\ S_1 &= (1-|\lambda|)^2+r^2-2r(1-|\lambda|)\cos(\theta-t),\\ S_2 &= 2|\lambda|+2r\cos(\theta-t)-2\mathrm{Re}\,\bar{w}z.\end{aligned}$$

For S_1 one already has $S_1<2(1-|\lambda|)^2$. To get

$$S_2 = 2|\lambda|\,(1-\cos(\theta-\varphi))+2r\,(\cos(\theta-t)-\cos(t-\varphi)) = O\left((1-|\lambda|)^2\right),$$

we choose w in the following way. If $|\theta|\ge t(M\gamma/2)$, we put $\varphi=\theta$, so $w=\lambda/|\lambda|$ and $S_2=0$. If $|\theta|<t(M\gamma/2)$, we put

$$\varphi = \begin{cases} t(M\gamma/2), & 0\le\theta<t(M\gamma/2),\\ -t(M\gamma/2), & -t(M\gamma/2)<\theta<0.\end{cases}$$

Then, by (1.1), $|\varphi-\theta|<t(M\gamma/2)\le\pi M\gamma/4$. On the other hand, by the hypothesis of lemma

$$\begin{aligned}M\gamma\le|1-\lambda| &= \left|1-|\lambda|+|\lambda|(1-e^{i\theta})\right|\\ &\le 1-|\lambda|+|\theta|\le 1-|\lambda|+\frac{\pi M\gamma}{4},\end{aligned}$$

so $1-|\lambda|\ge M\gamma(1-\pi/4)$ and $|\varphi-\theta|\le\frac{\pi}{4-\pi}(1-|\lambda|)$. Hence

$$\begin{aligned}S_2 &\le 4\sin^2\frac{\theta-\varphi}{2}+4r\left|\sin\frac{\theta-\varphi}{2}\sin\frac{\theta+\varphi-2t}{2}\right|\\ &\le\left(\frac{\pi}{4-\pi}\right)^2(1-|\lambda|)^2+\frac{\pi}{5(4-\pi)}(1-|\lambda|)^2<26(1-|\lambda|)^2\end{aligned}$$

and $|w-z|^2<28(1-|\lambda|)^2$.

Note also that in both cases above we have $|\varphi|\ge t(M\gamma/2)$, that is

$$|1-w|\ge\frac{M\gamma}{2}. \tag{1.8}$$

To fix the second term in (1.7) we take $0<\varepsilon<(1-|\lambda|)^2$, so

$$\varepsilon^2+2\varepsilon\mathrm{Re}\,(1-\bar{w}z)\le 5\varepsilon<5(1-|\lambda|)^2,$$

and eventually

$$|(1+\varepsilon)w-z|^2<31(1-|\lambda|)^2.$$

It follows now from the above inequality and (1.6) that W_λ admits the lower bound

$$W_\lambda(z)>\frac{1/3(1-|\lambda|)}{31(1-|\lambda|)^2}>\frac{1}{100}\frac{1}{1-|\lambda|},$$

where $z \in \partial B(\lambda, r)$. So,

$$\log \frac{1}{|\lambda|} W_\lambda(z) > \frac{1}{100}$$

for these z. By the Maximum Principle $V_\lambda < 500\, W_\lambda$ on U, and by letting $\varepsilon \to 0$ we obtain

$$\log \frac{1}{|b_\lambda(z)|} < 500 \log \frac{1}{|\lambda|} \frac{1-|z|^2}{|w-z|^2},$$

where $z \in U$.

Note that by the assumption of the lemma $|1-\lambda| = l \geq M\gamma$, so $1-|\lambda| \leq |1-\lambda| = l$, and if $z \in \partial B(\lambda, r)$, then

$$|1-\lambda - re^{it}| \geq l - \frac{1-|\lambda|}{4} \geq \frac{3}{4} M\gamma > 2\gamma,$$

which means that the arc $\{|1-z| = \gamma,\ |z|<1\}$ lies inside U. For such z by (1.8) $|w-z| \geq |1-w| - |1-z| \geq \big(M/2-1\big)\gamma$, so

$$\frac{1-|z|^2}{|w-z|^2} < \frac{2}{\big(M/2-1\big)^2\gamma}\,.$$

The proof is complete. $\square$

An invariant form of (1.5) is

$$\log \frac{1}{|b_\lambda(z)|} \leq \frac{1}{4N\gamma} \log \frac{1}{|\lambda|}\,,$$

for any $\zeta \in \mathbb{T}$ with the properties $|\zeta - z| = \gamma,\ |\zeta - \lambda| \geq M\gamma$.

We decompose the unit disk into a union of disjoint sets

$$\mathbb{D} = \Omega \bigcup \left(\bigcup_{n,k} \Omega_{n,k} \right), \quad \Omega_{n,k} = \{z \in \mathbb{D} : 2^{-n-1} < |z - \zeta_k| \leq 2^{-n}\},$$

where $k = 1, 2, \dots, N,\ n = L, L+1, \dots$. Here $L = L(E)$ is a large parameter, so that

$$\text{dist}\,(\Omega_{L,k}, \zeta_s) \geq 2^{-L}, \tag{1.9}$$

where $s \neq k,\ k = 1, \dots, N$. The latter obviously yields the same inequality for $\text{dist}\,(\Omega_{n,k}, \zeta_s)$ for all $n \geq L$.

Let us fix a pair (n,k), and define numbers

$$\gamma_s = \begin{cases} 2^{-L-1}/M, & s \neq k, \\ 2^{-n-1}/M, & s = k, \end{cases}$$

where $s = 1, \dots, N$, and M is from Lemma 1.1. Now (1.9) reads

$$\text{dist}\,(\Omega_{n,k}, \zeta_s) \geq M\gamma_s. \tag{1.10}$$

Define three sets of arcs

$$\Gamma_s := \partial B(\zeta_s, \gamma_s) \bigcap \mathbb{D}, \qquad \tilde{\Gamma}_s := \{\zeta \in \mathbb{T} : |\zeta - \zeta_s| \leq \gamma_s\},$$

the arcs $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_{s+1}$ are separated by $\hat{\Gamma}_s \subset \mathbb{T}$, $s = 1, \dots, N$. Set $\Delta_{n,k} \subset \mathbb{D}$ in a way that

$$\partial \Delta_{n,k} = \left(\bigcup_{s=1}^{N} \Gamma_s \right) \bigcup \left(\bigcup_{s=1}^{N} \hat{\Gamma}_s \right).$$

Let $\{z_j\}_{j=1}^m$ be a finite number of zeros of f (counting multiplicity) in $\Omega_{n,k}$, and

$$b_j := b_{z_j}(z) = \frac{z - z_j}{1 - \bar{z}_j z},$$

for $j = 1, \dots, m$. Consider the functions

$$g_s(z) = g_{\gamma_s}^{\rho_s}(z\bar{\zeta}_s), \qquad \rho_s = \frac{4D}{\gamma_s^q},$$

$$g_{j,s}(z) = g_{\gamma_s}^{\rho_{j,s}}(z\bar{\zeta}_s), \qquad \rho_{j,s} = \frac{1}{N\gamma_s} \log \frac{1}{|z_j|},$$

with $s = 1, \dots, N$, $j = 1, \dots, m$. Recall that g_s are defined in (1.3). For $z \in \Gamma_s$ we have

$$\log |b_j(z) g_{j,s}(z)| = \log |b_j(z)| + \rho_{j,s} \log |g_{\gamma_s}(z\bar{\zeta}_s)|.$$

By (1.10) we can apply Lemma 1.1, which along with (1.2) gives

$$\log |b_j(z) g_{j,s}(z)| \geq \frac{1}{4N\gamma_s} \log |z_j| + \frac{1}{4N\gamma_s} \log \frac{1}{|z_j|} = 0,$$

so

$$|b_j(z) g_{j,s}(z)| \geq 1, \tag{1.11}$$

for $z \in \Gamma_s$, $j = 1, \dots, m$.

Note also, that, for $z \in \mathbb{D}$ (see (1.4))

$$|b_j(z) g_{j,s}(z)| \leq e.$$

Next, by (1.2)

$$\log |g_s(z)| = \rho_s \log |g_{\gamma_s}(z\bar{\zeta}_s)| \geq \frac{D}{\gamma_s^q} = \frac{D}{|z - \zeta_s|^q}, \tag{1.12}$$

where $z \in \Gamma_s$.

Lemma 1.2. *A function*

$$F(z) := \frac{f(z)}{\prod_{j=1}^m b_j(z) \hat{g}_j(z) \prod_{l=1}^N g_l(z)}, \quad \hat{g}_j(z) := \prod_{l=1}^{N} g_{j,l}(z) \tag{1.13}$$

is analytic on $\mathbb{D}$, and satisfies

$$\log |F(z)| \leq 0, \tag{1.14}$$

for $z \in \bigcup_{l=1}^N \Gamma_l$,

$$\log |F(\zeta)| \leq \frac{D}{\operatorname{dist}(\zeta, E)^q}, \tag{1.15}$$

for $\zeta \in \bigcup_{l=1}^N \hat\Gamma_l$. *Furthermore,*

$$\log|F(0)| \ge \frac{1}{2}\sum_{j=1}^m \log\frac{1}{|z_j|} - 2D\sum_{l=1}^N \gamma_l^{1-q}. \tag{1.16}$$

Proof. For $z \in \Gamma_s$, $s = 1, \dots, N$, we have

$$\log|f(z)| \le \frac{D}{\mathrm{dist}(z,E)^q} = \frac{D}{|z-\zeta_s|^q},$$

and $|b_j| = |\hat g_j| = |g_s| = 1$ for the rest of the boundary of $\Delta_{n,k}$. So bounds (1.14) and (1.15) follow immediately from definition (1.13).

To prove (1.16), we write

$$\log|F(0)| = -\sum_{j=1}^m \Big(\log|b_j(0)| + \log|\hat g_j(0)|\Big) - \sum_{l=1}^N \log|g_l(0)|.$$

Apply the bound for the harmonic measure (see (1.1))

$$\log|g_l(0)| \le \frac{4D}{\gamma_l^q}\frac{\gamma_l}{2} = 2D\gamma_l^{1-q},$$

$$\log|\hat g_j(0)| = \sum_{l=1}^N \rho_{j,l}\log|g_{\gamma_l}(0)| \le \frac{1}{N}\sum_{l=1}^N \frac{1}{\gamma_l}\log\frac{1}{|z_j|}\frac{\gamma_l}{2} = \frac{1}{2}\log\frac{1}{|z_j|},$$

so

$$\log|b_j(0)| + \log|\hat g_j(0)| \le \log|z_j| + \frac{1}{2}\log\frac{1}{|z_j|} = -\frac{1}{2}\log\frac{1}{|z_j|}.$$

The proof of the lemma is complete. □

Proof of Theorem 0.1. Define an outer function F^* in $\mathbb{D}$ by its boundary values

$$|F^*(\zeta)| = \begin{cases} 1, & \zeta \in \bigcup_{l=1}^N \tilde\Gamma_l, \\ \exp\Big(\dfrac{D}{\mathrm{dist}(\zeta,E)^q}\Big), & \zeta \in \bigcup_{l=1}^N \hat\Gamma_l. \end{cases}$$

As $|F^*| \ge 1$ on $\mathbb{T}$, then $|F^*| \ge 1$ in the whole disk, so by Lemma 1.2 $|F| \le |F^*|$ on $\partial\Delta_{n,k}$, and hence by the Maximum Modulus Principle $|F| \le |F^*|$ in $\Delta_{n,k}$. In particular,

$$\log|F(0)| \le \log|F^*(0)|.$$

The upper bound for the RHS follows from

$$\log|F^*(0)| = \int_{\mathbb{T}} \log|F^*(\zeta)|\,dm = D\sum_{l=1}^N \int_{\hat\Gamma_l} \mathrm{dist}(\zeta,E)^{-q}\,dm.$$

It is easy to estimate a typical integral in the above sum (note that by the definition $\gamma_k \le \gamma_l$ for all $l = 1, \dots, N$)

$$\int_{\hat\Gamma_l} \mathrm{dist}(\zeta,E)^{-q}\,dm \le \begin{cases} \frac{2\pi^{q-1}}{q-1}\left(\frac{1}{\gamma_k}\right)^{q-1}, & q > 1, \\ \log\frac{1}{\gamma_k}, & q = 1. \end{cases}$$

Hence for $q>1$ (for $q=1$ the argument is the same)

$$\log|F^*(0)| \le C(q)ND\left(\frac{1}{\gamma_k}\right)^{q-1},$$

so by (1.16)

$$\sum_{j=1}^m (1-|z_j|) \le \sum_{j=1}^m \log\frac{1}{|z_j|} < C(q,N)D\left(\frac{1}{\gamma_k}\right)^{q-1}. \tag{1.17}$$

By the definition of γ_k and $\Omega_{n,k}$ we have

$$2M\gamma_k = 2^{-n} \ge |z_j-\zeta_k| = \operatorname{dist}(z_j,E),$$

so (1.17) implies

$$\begin{aligned}\sum_{j=1}^m (1-|z_j|)\operatorname{dist}(z_j,E)^{q-1+\varepsilon} &\le \sum_{j=1}^m (1-|z_j|)(2M\gamma_k)^{q-1+\varepsilon}\\ &\le CD(2M\gamma_k)^{\varepsilon} = CD2^{-n\varepsilon}.\end{aligned}$$

Summation over $n\ge L$, and then over $k=1,\dots,N$ gives

$$\sum_{z_j\in\Omega} (1-|z_j|)\operatorname{dist}(z_j,E)^{q-1+\varepsilon} \le CD,$$

where $\Omega := \bigcup_{n,k}\Omega_{n,k}$.

The same reasoning with $\gamma_s = 1/2^L M$, $s=1,\dots,N$ applies to the domain $\Omega_0 := \{z\in\mathbb{D} : |z-\zeta_k| > 2^{-L},\ k=1,\dots,N\}$. The proof is complete. □

Proof of Theorem 0.2. Denote $\tau_n := 1-2^{-n}$, $n=0,1\dots$ and put $f_n(z) := f(\tau_n z)$. Then

$$|f_n(z)| \le \exp\Bigl(\frac{D_1}{(1-|\tau_n z|)^p \operatorname{dist}(\tau_n z,E)^q}\Bigr).$$

An elementary inequality

$$\frac{1-|z|}{1-\tau|z|} \le \frac{|z-\zeta|}{|\tau z-\zeta|} \le \frac{1+|z|}{1+\tau|z|} < 2, \tag{1.18}$$

which holds for $z\in\mathbb{D}$, $\zeta\in\mathbb{T}$ and $0\le\tau<1$, gives

$$\operatorname{dist}(\tau_n z,E) > \frac{1}{2}\operatorname{dist}(z,E)$$

so

$$|f_n(z)| \le \exp\Bigl(\frac{D_2}{\operatorname{dist}(z,E)^q}\Bigr),$$

where $D_2 = 2^{np+q}D_1$. If $Z_f=\{z_j\}$, $Z_{f_n}=\{z_{j,n}\}$ then

$$z_{j,n} = \frac{z_j}{\tau_n} :\ |z_j| < \tau_n.$$

By Theorem 0.1

$$\sum_{z_j:|z_j|<\tau_n}\Big(1-\frac{|z_j|}{\tau_n}\Big)\,\mathrm{dist}\Big(\frac{z_j}{\tau_n},E\Big)^r \le C(\varepsilon,q,E)2^{np+q}D_1, \tag{1.19}$$

$r=(q-1+\varepsilon)_+$. We readily continue as

$$\begin{aligned}\text{LHS of (1.19)} &\ge \sum_{\tau_{n-2}\le|z_j|<\tau_{n-1}}\Big(1-\frac{|z_j|}{\tau_n}\Big)\,\mathrm{dist}\Big(\frac{z_j}{\tau_n},E\Big)^r\\ &\ge \frac{1}{4^{r+1}}\sum_{\tau_{n-2}\le|z_j|<\tau_{n-1}}(1-|z_j|)\,\mathrm{dist}(z_j,E)^r,\end{aligned}$$

and, consequently,

$$2^{-n(p+\varepsilon)}\sum_{\tau_{n-2}\le|z_j|<\tau_{n-1}}(1-|z_j|)\,\mathrm{dist}(z_j,E)^r \le C2^{-n\varepsilon}D_1.$$

Since $1-|z_j|\le 1-\tau_{n-2}$ then

$$2^{-n(p+\varepsilon)}\ge\frac{1}{4^{p+\varepsilon}}(1-|z_j|)^{p+\varepsilon},$$

and finally

$$\sum_{\tau_{n-2}\le|z_j|<\tau_{n-1}}(1-|z_j|)^{p+\varepsilon+1}\,\mathrm{dist}(z_j,E)^r \le C2^{-n\varepsilon}D_1.$$

It remains only to sum up over n from 2 to ∞. □

2. Applications to complex Jacobi matrices

We are interested in complex-valued Jacobi matrices of the form

$$J=J(\{a_k\},\{b_k\},\{c_k\})=\begin{bmatrix} b_1 & c_1 & 0 & \dots\\ a_1 & b_2 & c_2 & \dots\\ 0 & a_2 & b_3 & \dots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix} \tag{2.1}$$

where $a_k,b_k,c_k\in\mathbb{C}$. We assume J to be a compact perturbation of the free Jacobi matrix $J_0=J(\{1\},\{0\},\{1\})$, or, equivalently, $\lim_{k\to+\infty}a_k=\lim_{k\to+\infty}c_k=1$, $\lim_{k\to+\infty}b_k=0$. It is well-known that in this situation $\sigma_{ess}(J)=[-2,2]$. The point spectrum of J is denoted by $\sigma_p(J)$; the eigenvalues $\lambda\in\sigma_p(J)$ have finite algebraic (and geometric) multiplicity, and the set of their limit points lies on the interval [-2,2] (see, e.g., [5, Lemma I.5.2]).

The structure of $\sigma_p(J)$ and, especially, its behavior near $[-2,2]$, is an important part of the spectral analysis of complex Jacobi matrices. We quote a theorem from Golinskii-Kupin [6] to give a flavor of the known results.

Theorem 2.1. *For $p \ge 1$,*

$$\begin{aligned}
\sum_{\lambda\in\sigma_p(J)} (\operatorname{Re}\lambda - 2)_+^p \quad &+ \sum_{\lambda\in\sigma_p(J)} (\operatorname{Re}\lambda + 2)_-^p \\
&\le c_p\Big(\sum_{k=1}^{\infty} |\operatorname{Re} b_k|^{p+1/2} + 4\Big|\frac{a_k+\bar c_k}{2} - 1\Big|^{p+1/2}\Big), \\
\sum_{\lambda\in\sigma_p(J)} (\operatorname{Re}\lambda - 2)_+^p \quad &+ \sum_{\lambda\in\sigma_p(J)} (\operatorname{Re}\lambda + 2)_-^p \\
&\le 3^{p-1}\Big(\sum_{k=1}^{\infty} |\operatorname{Re} b_k|^{p} + 4\Big|\frac{a_k+\bar c_k}{2} - 1\Big|^{p}\Big),
\end{aligned}$$

where $x_+ = \max\{x,0\}$, $x_- = -\min\{x,0\}$, and

$$c_p = \frac{3^{p-1/2}}{2}\frac{\Gamma(p+1)}{\Gamma(p+3/2)}\frac{\Gamma(2)}{\Gamma(3/2)}. \tag{2.2}$$

To formulate the results of this section, we use the Schatten-von Neumann classes of compact operators $\mathcal{S}_p$, $p \ge 1$, and regularized determinants $\det_p(I + A)$, $A \in \mathcal{S}_p$. An extensive information on the subject is in Gohberg-Krein [5, Ch. 3, 4], Simon [11, Ch. 2, 9]. The norms in $\mathcal{S}_p$, $1 \le p < \infty$, $\mathcal{S}_\infty$, are denoted by $\|\cdot\|_p$, $\|\cdot\|$, respectively.

Let $\lambda \in \hat{\mathbb{C}}\backslash[-2,2]$. We map this domain onto $\mathbb{D}$ in the standard way $\lambda = z + 1/z$, $z = \frac{1}{2}(\lambda - \sqrt{\lambda^2-4})$, $z \in \mathbb{D}$. Here is a list of elementary properties of the above-mentioned concepts and their connections to the spectral characteristics of operator J. Suppose $J - J_0 \in \mathcal{S}_p$, $1 \le p < \infty$.

- $(J-J_0)(J_0-\lambda)^{-1} \in \mathcal{S}_p$.
- for an integer $p \ge 1$,

$$u_p(\lambda) = \det{}_p(J-\lambda)(J_0-\lambda)^{-1} = \det{}_p(I + (J-J_0)(J_0-\lambda)^{-1}),$$

  and so the regularized perturbation determinant u_p is well-defined.
- the function u_p is analytic on $\hat{\mathbb{C}}\backslash[-2,2]$. Furthermore, $Z_{u_p} = \sigma_p(J)$ taking into account the multiplicities, i.e., the order of a zero λ_0 of the function u_p is equal to the total multiplicity of $\lambda_0 \in \sigma_p(J)$.
- We also have

$$\begin{aligned}
|\det{}_p(J-\lambda)(J_0-\lambda)^{-1}| \;&\le\; \exp\Big(\frac{1}{p}\|(J-J_0)(J_0-\lambda)^{-1}\|_p^p\Big) \\
&\le\; \exp\Big(\frac{1}{p}\|(J-J_0)\|_p^p\,\|(J_0-\lambda)^{-1}\|^p\Big) \\
&=\; \exp\Big(\frac{1}{p}\|(J-J_0)\|_p^p\,\mathrm{dist}\,(\lambda,[-2,2])^{-p}\Big).
\end{aligned}$$

The first inequality is in [11, Ch.9]. Then we recall that $\mathcal{S}_p$ is an ideal with respect to the multiplication and use the expression for the resolvent $\|(J_0-\lambda)^{-1}\|$ of a *self-adjoint* operator J_0.

Lemma 2.2. *Let* $\lambda = z+1/z$, $z\in\mathbb{D}\backslash\mathbb{D}_\delta$, *where* $0<\delta<1$. *Then*

$$\begin{aligned} \text{dist}\,(\lambda,[-2,2]) &\asymp (1-|z|)|1-z^2|, \\ |1\pm z|^2 &\asymp |\lambda\pm 2|, \quad 1-|z|\asymp\frac{\text{dist}\,(\lambda,[-2,2])}{|\lambda^2-4|^{1/2}}. \end{aligned}$$

For the proof of the first relation see [10, p.9, Corollary 1.4].

Let $f_p(z)=u_p(\lambda(z))$, $p>1$. Then

$$|f_p(z)|\le \exp\Big(\frac{\|J-J_0\|_p^p}{p(1-|z|)^p|1-z^2|^p}\Big).$$

For $p=1$, it is proved in [2] that

$$|f_1(z)|\le\frac{2\|J-J_0\|_1}{|1-z^2|}\exp\Big(\frac{2\|J-J_0\|_1}{|1-z^2|}\Big).$$

Theorem 2.3. *For* $p=1$ *and every* $\varepsilon>0$ *we have*

$$\sum_{\lambda\in\sigma_p(J)}\frac{\text{dist}\,(\lambda,[-2,2])}{|\lambda^2-4|^{(1-\varepsilon)/2}}\le C(\varepsilon,\|J-J_0\|,p)\|J-J_0\|_1. \tag{2.3}$$

For integer $p\ge 2$ *and every* $\varepsilon>0$ *we have*

$$\sum_{\lambda\in\sigma_p(J)}\frac{\text{dist}\,(\lambda,[-2,2])^{p+1+\varepsilon}}{|\lambda^2-4|}\le C(\varepsilon,\|J-J_0\|,p)\|J-J_0\|_p^p. \tag{2.4}$$

Proof. Indeed, $\sigma_p(J)$ is in $B(0,2+\|J-J_0\|)$, so Z_{f_p} lies in $\mathbb{D}\backslash\mathbb{D}_\delta$ with $0<\delta<1$ depending on $\|J-J_0\|$.

For instance, to get the second relation we apply Theorem 0.1 to f_p:

$$\begin{aligned} &\sum_{\lambda\in\sigma_p(J)}(1-|z(\lambda)|)^2\text{dist}\,(\lambda,[-2,2])^{p-1+\varepsilon} \\ &\quad=\sum_{z\in Z_{f_p}}(1-|z|)^{p+1+\varepsilon}|1-z^2|^{p-1+\varepsilon}\le C(\varepsilon,\|J-J_0\|,p)\|J-J_0\|_p^p. \end{aligned}$$

It remains to use Lemma 2.2. The first relation follows in a similar way from Theorem 0.2. □

It is worth mentioning that the only ingredient we need to make the proof of Theorem 2.3 work, is the bound on the resolvent of the unperturbed operator. Neither its self-adjoint property, nor three-diagonal form are required. For instance, we can prove the same result for an operator, similar to J_0 and its perturbations.

Although we do not claim the results being optimal, the bounds give a lot of interesting information. As compared to classical Lieb-Thirring inequalities for complex Jacobi matrices, the bounds (2.3) and (2.4)

involve the whole point spectrum $\sigma_p(J)$, and not of its relatively simple parts (see [6]). In particular, we see that $\sigma_p(J)$ behaves differently along the interval $(-2,2)$ and in the neighborhoods of its endpoints ± 2.

An analogous theorem holds, of course, for multidimensional Jacobi matrices, see [6, Sect. 2] for definitions.

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Université Aix-Marseille, 39, rue Joliot-Curie, 13453 Marseille, Cedex 13, France

E-mail address: borichev@cmi.univ-mrs.fr, kupin@cmi.univ-mrs.fr

Mathematics Division, Institute for Low Temperature Physics and Engineering, 47 Lenin ave., Kharkov 61103, Ukraine

E-mail address: leonid.golinskii@gmail.com